Reliability Analysis of Communication Networks

Mathematical models and algorithms

Peter Tittmann

Hochschule Mittweida

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Models for Communication Networks

Communication network	**Mathematical model**
Network topology	(un-) directed graph
Server Switch Base station	Vertex
Fiber optical transmission line Transmission line (copper) Wireless channel	Edge (arc)
Server failure probability	Weights of vertices
Link availability Line length (costs) Transfer rate	Edge weights
User terminal PoP	Terminal vertex

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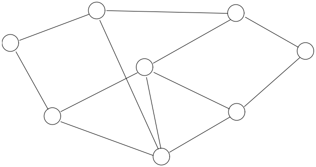
- What is the probability of packet loss?
- How is traffic load distributed over the network?

Constraints

- Network technology, used protocol (IP, ATM, SS7, GSM)
- Routing, redundancy properties
- Service under consideration
- Supply of data
- Type of failure

Directed or undirected graph $G = (V, E)$



p_e … availability of edge $e \in E$

All edges are assumed to fail independently.

$\mathcal{W}$	set of paths
$\mathcal{C}$	set of cuts

Reliability Measures

- All-terminal reliability (connectedness probability)

$$R(G) = P(\{G \text{ is connected}\})$$

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- Probability of a given data transfer between two vertices.
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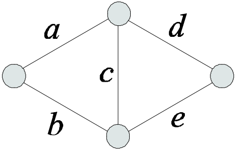
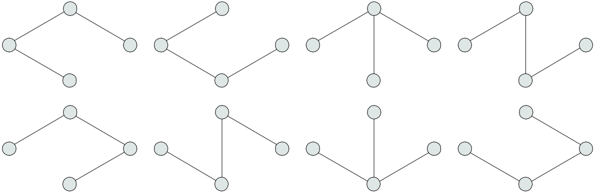
- Importance measures

All-Terminal Reliability
Example



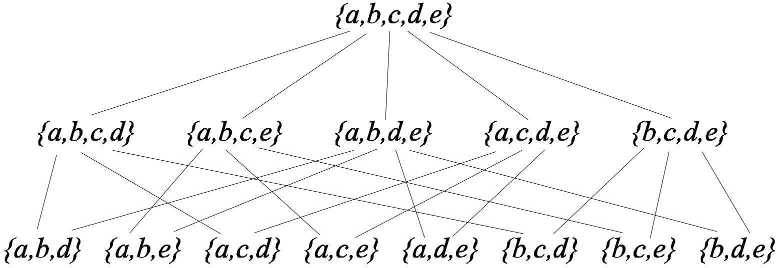
Sets of minimum paths – spanning trees
$\mathcal{W} = \{\{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{b, c, d\}, \{b, c, e\},$



All-Terminal Reliability

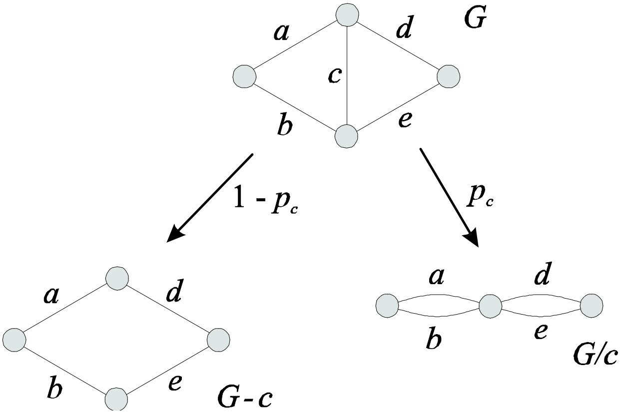
Example

$$
\begin{aligned}
R(G) = \; & p_a p_b p_d + p_a p_b p_e + p_a p_c p_d + p_a p_c p_e + p_a p_d p_e + p_b p_c p_d \\
& + p_b p_c p_e + p_b p_d p_e - 2 p_a p_b p_c p_d - 2 p_a p_b p_c p_e - 3 p_a p_b p_d p_e \\
& - 2 p_a p_c p_d p_e - 2 p_b p_c p_d p_e + 4 p_a p_b p_c p_d p_e
\end{aligned}
$$

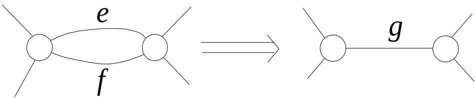


Decomposition

$$R(G) = (1 - p_c)\, R(G - c) + p_c R(G/c)$$

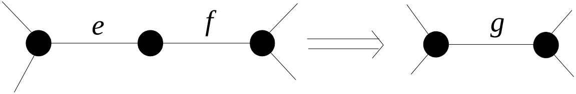


1. Parallel reduction



$$p_g = p_e + p_f - p_e p_f$$

2. Degree-2-reduction



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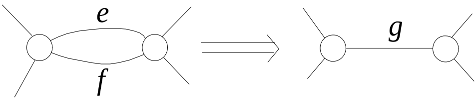

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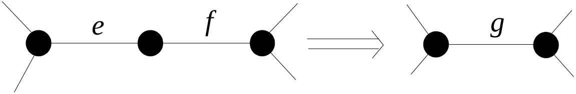
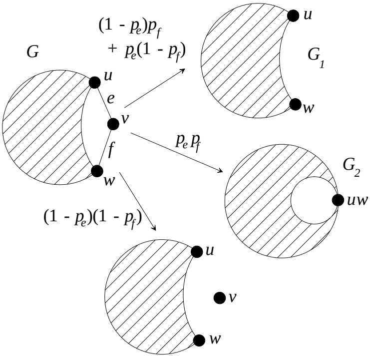


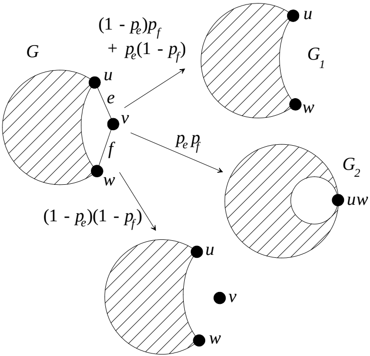
$$\omega = p_e + p_f - p_e p_f, \qquad p_g = \frac{p_e p_f}{p_e + p_f - p_e p_f}. \qquad R(G) = \omega R(G')$$

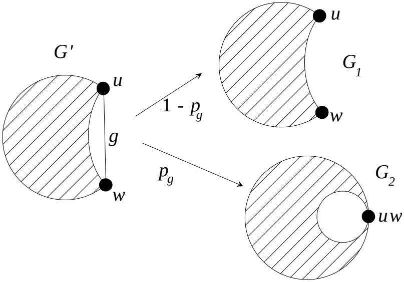


$$R(G) = (p_e + p_f - 2p_e p_f)\,R(G_1) + p_e p_f\,R(G_2)$$

All-Terminal Reliability

Reductions





$$R\left(G'\right) = \left(1 - p_g\right) R\left(G_1\right) + p_g R\left(G_2\right)$$

$$R\left(G\right) = \left(p_e + p_f - 2 p_e p_f\right) R\left(G_1\right) + p_e p_f R\left(G_2\right)$$

Reduction principle

$$
\begin{aligned}
R(G) &= \omega R(G') \\
R(G) &= (p_e + p_f - 2p_e p_f)\, R(G_1) + p_e p_f\, R(G_2) \\
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Method of coefficient comparison

$$
\begin{aligned}
p_e + p_f - 2p_e p_f &= \omega (1 - p_g) \\
p_e p_f &= \omega p_g
\end{aligned}
$$

Reduction principle

$$\begin{aligned} R(G) &= \omega R(G') \\ R(G) &= (p_e + p_f - 2p_e p_f) R(G_1) + p_e p_f R(G_2) \\ R(G') &= (1 - p_g) R(G_1) + p_g R(G_2) \end{aligned}$$

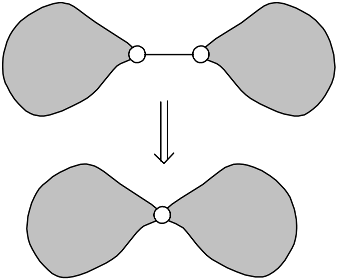
Method of coefficient comparison

$$\begin{aligned} p_e + p_f - 2p_e p_f &= \omega (1 - p_g) \\ p_e p_f &= \omega p_g \end{aligned}$$

Solution (reduction parameter)

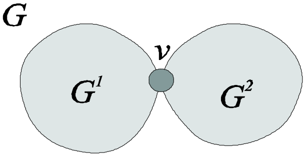
$$\begin{aligned} \omega &= p_e + p_f - p_e p_f \\ p_g &= \frac{p_e p_f}{p_e + p_f - p_e p_f} \end{aligned}$$

Bridge reduction



$$R(G) = p_e R\left(G'\right)$$

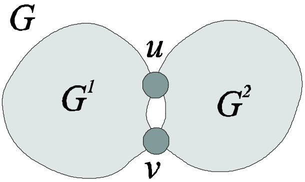
Cut vertex – articulation



$$\begin{aligned} G^1 \cup G^2 &= G \\ G^1 \cap G^2 &= (\{v\}, \emptyset) \end{aligned}$$

$$R(G) = R\left(G^1\right) R\left(G^2\right)$$

Separating vertex pair



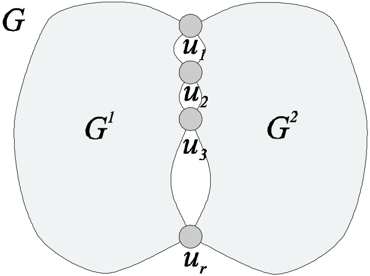
$$G^1 \cup G^2 = G$$
$$G^1 \cap G^2 = (\{u, v\}, \emptyset)$$

$$R(G) = R\left(G^1\right) R\left(G^2_{uv}\right) + R\left(G^1_{uv}\right) R\left(G^2\right) - R\left(G^1\right) R\left(G^2\right)$$

All-Terminal Reliability

Vertex Separators

U	separating vertex set
$\mathbb{P}(U)$	partition lattice of U
$P(G^1, \pi)$	probability that G^1 induces π
G^2_π	obtained from G^2 merging the blocks of π



$$R(G) = \sum_{\pi \in \mathbb{P}(U)} P(G^1, \pi)\, R(G^2_\pi)$$

Reliability polynomials

Definition

The reliability polynomial $R(G, p)$ is the probability that the undirected graph $G = (V, E)$ is connected, assuming all edges of G fail independently with probability $1 - p$.

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\begin{aligned}
R\left(G,p\right) &= \sum_{i=n-1}^{m} a_i p^i \\
&= \sum_{i=n-1}^{m} n_i p^i \left(1-p\right)^{m-i} \\
&= 1-\sum_{i=\lambda}^{m} c_i p^{m-i} \left(1-p\right)^i
\end{aligned}
$$

c_i ... number of cut sets of cardinality i

n_i ... number of spanning subgraphs of G

Reliability polynomials
Recursive Definition

$$R\left(G,p\right)=\begin{cases} p^{n-1}, \text{ if } G \text{ is a tree with } n \text{ vertices,} \\ 0, \text{ if } G \text{ is disconnected,} \\ pR\left(G/e\right)+\left(1-p\right)R\left(G-e\right), \text{ else.} \end{cases}$$

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Fact

This definition does not require any meaning of the variable p.

Reliability polynomials

Properties

1. If G is connected then
$$R(G,0) = 0,\ R(G,1) = 1$$
and
$$p^m \leq R(G,p) \leq 1 - (1-p)^m .$$

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$$p_1 < p_2 \ \Rightarrow\ R(G,p_1) < R(G,p_2)$$

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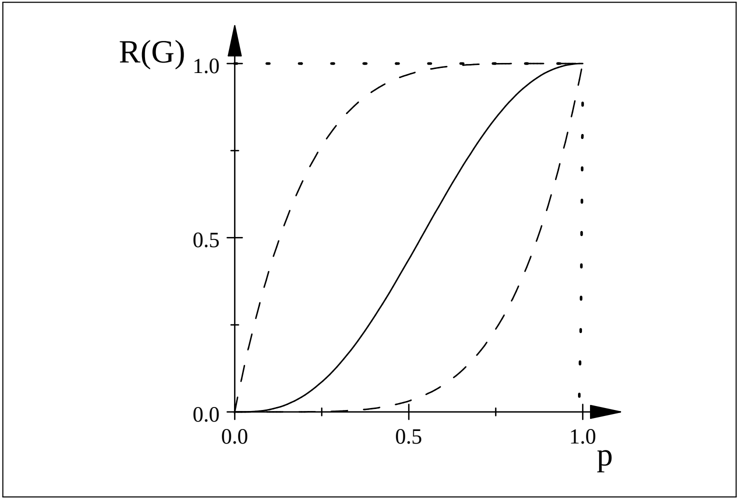
4. If G is biconnected then

$$\left.\frac{dR(G,p)}{dp}\right|_{p=1} = 0.$$

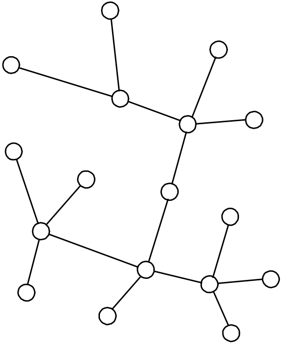
Reliability polynomials

Reliability Function



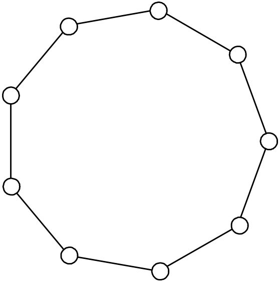
Trees

$$R(T_n, p) = p^{n-1}$$

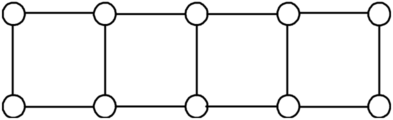


Cycles

$$R\left(C_n, p\right) = np^{n-1} - (n-1)\,p^n$$



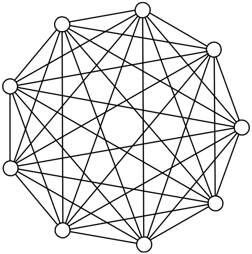
Ladder



$$R\left(L_n, p\right) = \frac{p^{2n-1}}{2^n \alpha}\left[\left(4 - 3p + \alpha\right)^n - \left(4 - 3p - \alpha\right)^n\right]$$

$$\text{with } \alpha = \sqrt{12 - 20p + 9p^2}$$

Complete graphs



- Recurrence equation $r_n := R\left(K_n, q\right), \; q := 1 - p$

$$\begin{aligned} r_n &= 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} r_k \\ r_1 &= 1 \end{aligned}$$

- Explicit representation

$$r_n = \sum_{\lambda \vdash n} (-1)^{|\lambda|+1} \binom{n}{\lambda} \binom{|\lambda|}{k} \frac{1}{|\lambda|} q^{a_2(\lambda)}$$

with

$$\lambda = (\lambda_1, \ ..., \ \lambda_k) = \left(1^{k_1} 2^{k_2} \cdots n^{k_n}\right)$$

and

$$a_2(\lambda_1, ..., \lambda_s) = \frac{1}{2}\left(n^2 - \sum_{i=1}^{s} \lambda_i^2\right)$$

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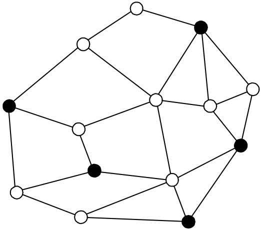
- Exponential generating function

$$r_n = q^{\frac{n^2}{2}} \left[\frac{z^n}{n!}\right] \ln\left(\sum_{n\geq 0} q^{-\frac{n^2}{2}} \frac{z^n}{n!}\right)$$

The K-terminal reliability



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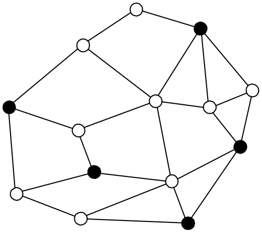
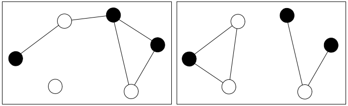


Definition

G is *K-connected* if all vertices of $K \subseteq V$ belong to one component of G.



The K-terminal reliability

$$R(G,K) = \begin{cases} 0 \text{ if } G \text{ is not } K\text{-connected}, \\ 1 \text{ if } G = (\{v\}, \emptyset),\ v \in K, \\ p_e R(G/e, K') + (1-p_e) R(G-e, K) \text{ else}, \end{cases}$$

where $K' = (K \setminus \{u,v\}) \cup X$, $e = \{u,v\}$, w is the vertex obtained merging vertices u and v and

$$X = \begin{cases} \{w\}, \text{ if } K \cap \{u,v\} \neq \emptyset \\ \emptyset, \text{ else}. \end{cases}.$$

Complete state enumeration:

$$R(G,K) = \sum_{F \subseteq E} \prod_{e \in F} p_e \prod_{e \in E \setminus F} (1-p)\, \xi(G[F], K)$$

with

$$\xi(G[F],K) = \begin{cases} 1, & \text{if } G[F] \text{ is } K\text{-connected,} \\ 0, & \text{else.} \end{cases}.$$

For $K = \{s, t\}$, the probability $R(G, K)$ is called *two-terminal reliability* (or *st*-reliability).

The K-terminal reliability

Properties

1. Let $\mathbf{p} = (p_1, ..., p_m)$ be the vector of edge availabilities of G. Then

$$\mathbf{p} \leq \mathbf{p}' \Rightarrow (G, K, \mathbf{p}) \leq R\left(G, K, \mathbf{p}'\right).$$

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$$K \subseteq L \Rightarrow R(G, K) \geq R(G, L)$$

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    1. $R(G, K \cup L) \geq R(G, K) + R(G, L) - 1,$

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   1. $R(G, K \cup L) \geq R(G, K) + R(G, L) - 1,$
   2. $R(G, K \cup L) \geq R(G, K)\, R(G, L).$

The K-terminal reliability

Properties

1. Let $\mathbf{p} = (p_1, ..., p_m)$ be the vector of edge availabilities of G. Then

$$\mathbf{p} \leq \mathbf{p}' \;\Rightarrow\; (G, K, \mathbf{p}) \leq R\left(G, K, \mathbf{p}'\right).$$

2. Monotonicity with respect to terminal vertex sets:

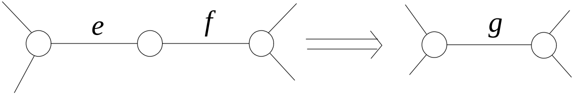
$$K \subseteq L \;\Rightarrow\; R(G, K) \geq R(G, L)$$

3. If $K \cap L \neq \emptyset$ then
   1. $R(G, K \cup L) \geq R(G, K) + R(G, L) - 1,$
   2. $R(G, K \cup L) \geq R(G, K)\, R(G, L)\,.$

4. The K-terminal reliability of the complete graph K_n:

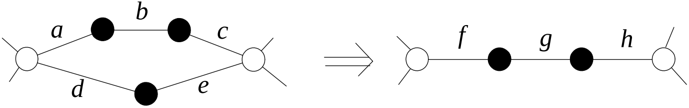
$$R(K_n, k) = \sum_{j=k}^{n} \binom{n-k}{j-k} r_j q^{j(n-j)}$$

Series reduction



$$p_g = p_e p_f$$

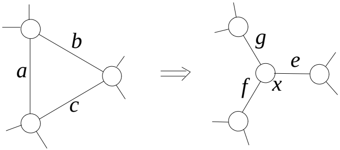
Polygon-to–chain reductions $R(G) = \omega R(G')$



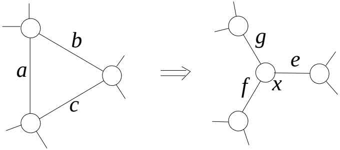
$$\alpha = ab(1-c)d(1-e), \ \beta = (1-a)bc(1-d)e$$
$$\gamma = a(1-b)c(d+e-2de) + (a+c-2ac)bd(1-e)$$
$$\delta = abcde\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)$$

$$f = \frac{\delta}{\beta+\delta}, \ g = \frac{\delta}{\gamma+\delta}, \ h = \frac{\delta}{\alpha+\delta},$$
$$\omega = \frac{(\alpha+\delta)(\beta+\delta)(\gamma+\delta)}{\delta^2}$$

Delta-Star reduction



Delta-Star reduction

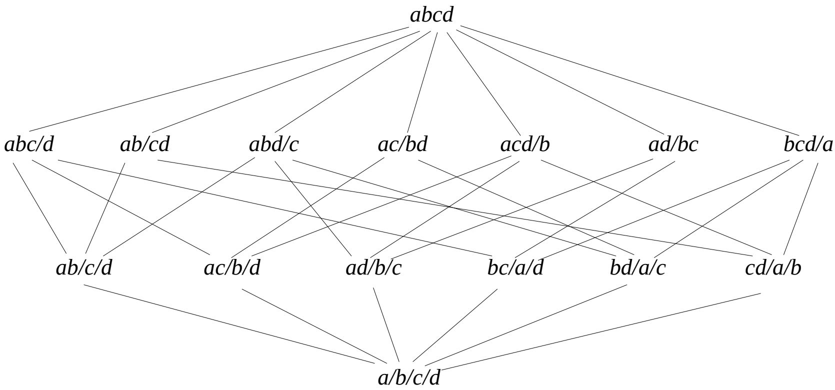


$$\begin{aligned}
\alpha &= a + b + c - ab - ac - cc + abc \\
\beta &= a + bc - abc \\
\gamma &= b + ac - abc \\
\delta &= c + ab - abc
\end{aligned}$$

$$e = \frac{\alpha}{\beta},\ f = \frac{\alpha}{\gamma},\ g = \frac{\alpha}{\delta},\ x = \frac{\beta\gamma\delta}{\alpha^2}$$

$M = \{a, b, c, d\}$, $\mathbb{P}(M)$ .. set of all partitions of M

$\pi \leq \sigma$ if and only if π is a *refinement* of σ



Möbius function

$$\mu\left(\pi,\sigma\right)=\left(-1\right)^{|\pi|-|\sigma|}\prod_{i=1}^{|\sigma|}\left(p_i-1\right)!$$

Supremum function

$$a\left(x,y\right)=\left[x\vee y=\hat{1}\right]$$

Inverse

$$a^{-1}\left(x,z\right)=\sum_{y}\frac{\mu\left(y,z\right)\mu\left(y,x\right)}{\mu\left(y,\hat{1}\right)}$$

Theorem

$$R(G,K) = \sum_{\sigma \geq \pi(K)} \sum_{\tau \in \mathbb{P}(V)} a^{-1}(\sigma,\tau) R(G_\tau)$$

Theorem

$$R(G,K) = \sum_{\sigma \geq \pi(K)} \sum_{\tau \in \mathbb{P}(V)} a^{-1}(\sigma,\tau)\, R(G_\tau)$$

Theorem

$$R(G) = \sum_{\sigma \in \mathbb{P}(V)} (-1)^{|\sigma|+1} (|\sigma|-1)!\, q^{|E(G,\sigma)|}$$

Tittmann, P.: *Partitions and network reliability*, Discrete Applied Mathematics 95 (1999), 445-453

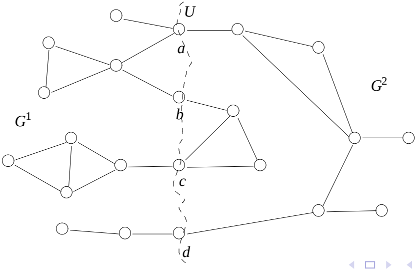
Separating Vertex Sets

Definition

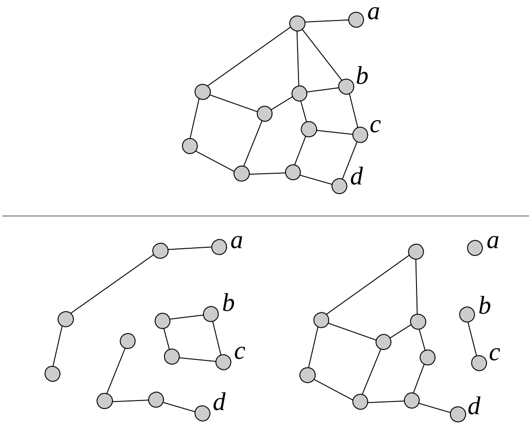
Let $G^1 = \left(V^1, E^1\right)$ and $G^2 = \left(V^2, E^2\right)$ be two subgraphs of $G = (V, E)$ such that

$$\begin{aligned} V^1 \cup V^2 &= V,\ V^1 \cap V^2 = U, \\ E^1 \cup E^2 &= E,\ E^1 \cap E^2 = \emptyset. \end{aligned}$$

Then U is called a *separating vertex set* of G.

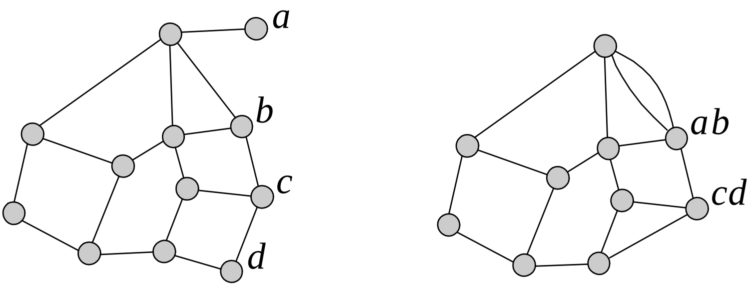


Separating Vertex Sets



induced partition: $\pi = a/bc/d$

Separating Vertex Sets



$$G \longrightarrow G_{ab/cd}$$

$$R\left(G\right)=\sum_{\pi\in\mathbb{P}(U)}P_{\pi}^{1}\,R\left(G_{\pi}^{2}\right)$$

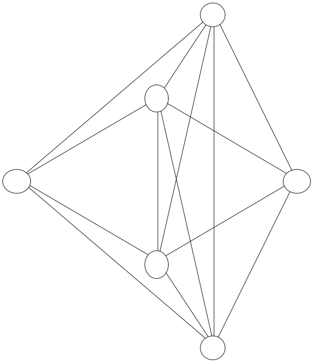
Number of terms:

$$B\left(u\right)\sim\frac{1}{\sqrt{u}}e^{u(r+1/r-1)-1}\text{ with }r\,e^{r}=u$$

Exponential generating function: $e^{e^{z}-1}=\sum_{n\geq 0}B(n)\frac{z^{n}}{n!}$

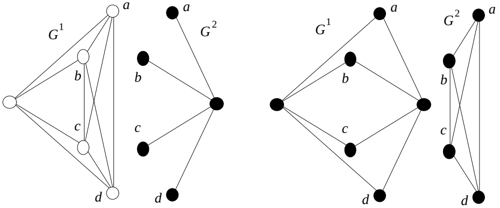
Separating Vertex Sets

Symmetric graphs



Separating Vertex Sets

Symmetric graphs



$$R(G) = \sum_{\lambda \vdash u} \binom{u}{\lambda} \binom{|\lambda|}{k} \frac{1}{|\lambda|} P_\lambda^1 R(G_\lambda^2)$$

Number of terms:

n	1	2	3	4	5	6	7	8	9
$B(n)$	1	2	5	15	52	203	877	4140	21147
$p(n)$	1	2	3	5	7	11	15	22	30
p_n	1	2	5	14	42	132	429	1430	4862
t_n	1	2	4	10	26	76	232	750	2494

A form of the splitting formula that requires only all-terminal reliability calculations for G^1 and G^2:

$$\boxed{R(G) = \sum_{\pi \in \mathbb{P}(U)} \sum_{\sigma \in \mathbb{P}(U)} R\left(G_\pi^1\right) a^{-1}(\pi, \sigma) R\left(G_\sigma^2\right)}$$

Special cases

$|U| = 1 : R(G) = R(G^1)\, R(G^2)$

$|U| = 2 : R(G) = R(G^1)\, R(G^2_{uv}) + R(G^1_{uv})\, R(G^2) - R(G^1)\, R(G^2)$

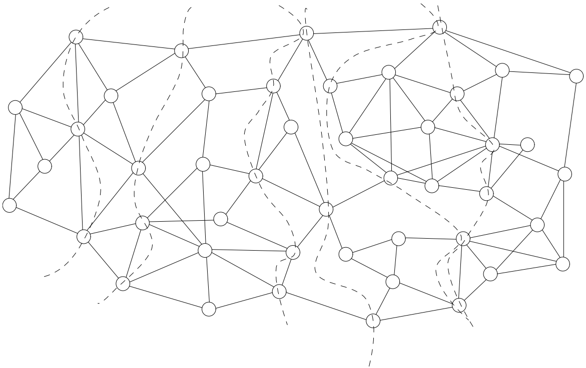
$|U| = 3 :$

$$A_3^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -1 \\ 2 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$$abc,\ ab/c,\ ac/b,\ bc/a,\ a/b/c$$

Separating Vertex Sets

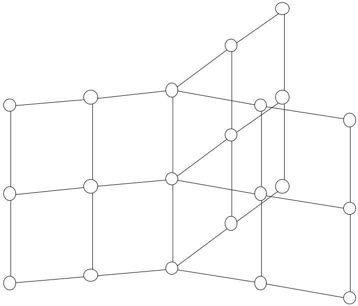
Multiple splitting



$$R\left(G_0 * G_1 * \ldots * G_n\right) = \mathbf{q}_0^T A_1^{-1} Q_1 A_2^{-1} Q_2 \cdots A_{n-1}^{-1} Q_{n-1} A_n^{-1} \mathbf{q}_n$$

Separating Vertex Sets

Multiple splitting



Gorlov, V.; Tittmann, P.: *A unified approach to the reliability of recurrent structures*, in Ellart von Collani et al. (editors): Advances in stochastic models for reliability, quality and safety, Birkhäuser, Boston, 1998

Tree decompositions

Definition

A *tree decomposition* of a graph $G = (V, E)$ is a pair (T, ϕ) with a tree $T = (W, F)$ and a map $\phi : W \to 2^V$ which assigns a vertex subset of G to each vertex of T such that

1. $\bigcup_{w \in W} \phi(w) = V$.
2. For each edge $\{u, v\} \in E$ there exists w in T such that $\{u, v\} \subseteq \phi(w)$.
3. Let $v \in W$ be on the path between u and w in T. Then we have $\phi(u) \cap \phi(w) \subseteq \phi(v)$.

Definition

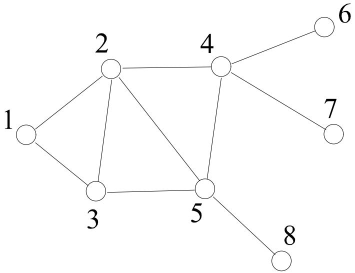
The *width* of a tree decomposition is

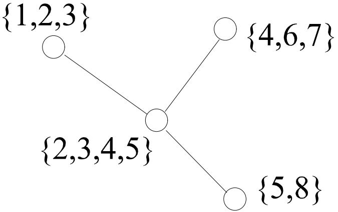
$$\max\{\phi(w) : w \in W\} - 1 \ .$$

The *treewidth* $tw(G)$ of G is the minimum width of a tree decomposition of G.

Tree decompositions

Example

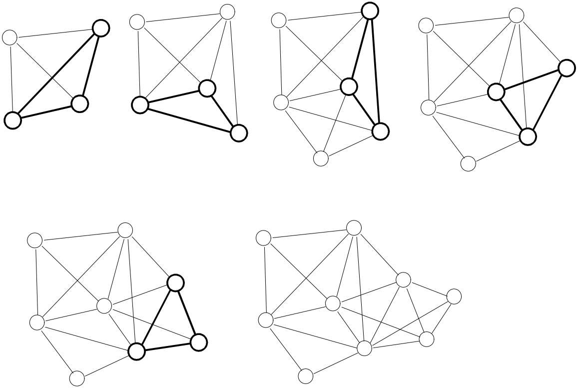




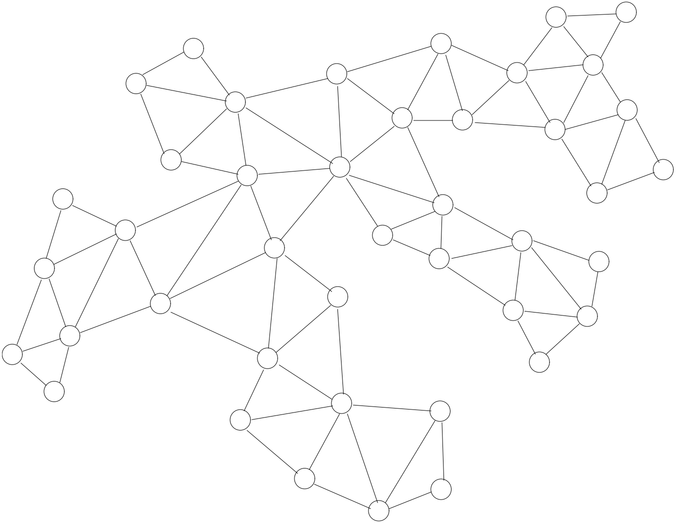
Partial k-trees

Definition

A graph $G = (V, E)$ is a k-tree if it is a complete graph with $k + 1$ vertices or if G contains a vertex $v \in V$ whose neighborhood induces a k-clique of G and $G - v$ is again a k-tree. A *partial k-tree* is a subgraph of a k-tree.



Partial k-trees



Path Decompositions

Definition

A *path decomposition* is a sequence

$$\mathcal{X} = (X_1, ..., X_r)$$

of vertex subsets of G such that the following conditions are satisfied

1. $\bigcup_{i=1}^{r} X_i = V$.
2. Each edge of G is contained in at least one of $X_1, ..., X_r$.
3. For all i, j, k with $1 \leq i < j < k \leq r$ the relation $X_i \cap X_k \subseteq X_j$ is valid.

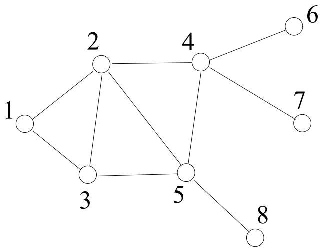
Path Decompositions

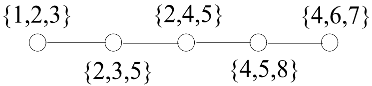
Definition

A *canonical path decomposition* satisfies in addition:

4. All subsets of the sequence $\mathcal{X}$ contain at most $pw\left(G\right)+1$ vertices.
5. $\left|X_{i+1}-X_i\right|=1$ for $i=1,\ldots,r-1$.

Path Decompositions





$$\{1\}, \{1,2\}, \{1,2,3\}, \{2,3\}, \{2,3,5\}, \{2,5\}, \{2,4,5\}, \{4,5\}, \{4,5,8\}, \{4,5\}, \{4\}, \{4,6\}, \{4\}, \{4,7\}, \{7\}$$

$$1, 2, 3, -1, 5, -3, 4, -2, 8, -8, -5, 6, -6, 7, -7$$

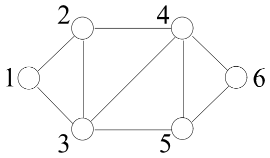
Path Decompositions

Definition

A *composition order* of $G = (V, E)$ is a sequence $s = (s_1, ..., s_r)$ of vertices and edges such that

1. The removal of all edges of s yields a canonical path decomposition of G.
2. Each edge is exactly once in s.
3. If $s_k = (u, v) \in E$ then there are indices i, j, p, q with $i, j < k$ and $p, q > k$ such that $u = s_i = s_p$ and $v = s_j = s_q$.

Path Decompositions



Composition order:

$$(1,\ 2,\ \{1,2\},\ 3,\ \{1,3\},\ -1,\ \{2,3\},\ 4,\ \{2,4\},\ -2,\ \{3,4\},\\ 5,\ \{3,5\},\ -3,\ \{4,5\},6,\ \{4,6\},\ -4,\ \{5,6\},\ -5,\ -6).$$

Assigned path decomposition:

$$(\emptyset,\ \{1\},\ \{1,2\},\ \{1,2,3\},\ \{2,3\},\{2,3,4\},\ \{3,4\},\ \{3,4,5\},\\ \{4,5\},\{4,5,6\},\ \{5,6\},\ \{6\},\ \emptyset)$$

State = (index, value)
Index: Partition of the set of active vertices
Value: Polynomial

Transformation of states

1. Vertex activation:

$$(\pi, P_\pi) \mapsto (\pi/\{v\}, P_\pi)$$

2. Vertex deactivation:
If $\{v\}$ is a singleton of π then $(\pi, P_\pi) \mapsto \emptyset$. Else:

$$(\pi, P_\pi) \mapsto \left(\pi - v, \sum_{\substack{v \in Y \in \pi \\ |Y| > 1}} P_\pi \right)$$

3. The insertion of an edge e generates two successor states:

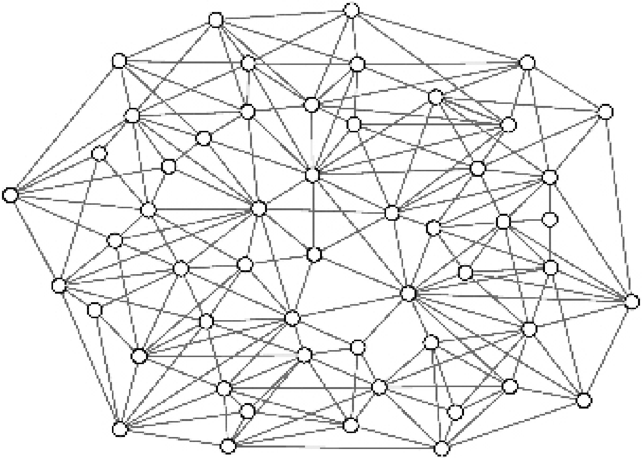
$$\text{(I)} \quad (\pi, P_\pi) \quad \mapsto \quad (\pi, (1-p)\, P_\pi)$$

$$\text{(II)} \quad (\pi, P_\pi) \quad \mapsto \quad (\pi \vee e,\; pP_\pi + P_{\pi \vee e})$$

s_i	X_{i+1}	Z_{i+1}
1	$\{1\}$	$\{(1,1)\}$
2	$\{1,2\}$	$\{(1/2,1)\}$
$\{1,2\}$	$\{1,2\}$	$\{(1/2,1-p),(12,p)\}$
3	$\{1,2,3\}$	$\{(1/2/3,1-p),(12/3,p)\}$
$\{1,3\}$	$\{1,2,3\}$	$\{\left(1/2/3,1-2p+p^2\right),$ $\left(12/3,p-p^2\right),$ $\left(13/2,p-p^2\right),123,p^2\}$
-1	$\{2,3\}$	$\{\left(2/3,2p-2p^2\right),\left(23,p^2\right)\}$
$\{2,3\}$	$\{2,3\}$	$\{\left(2/3,2p-4p^2+2p^3\right),$ $\left(23,3p^2-2p^3\right)\}$

Path Decompositions

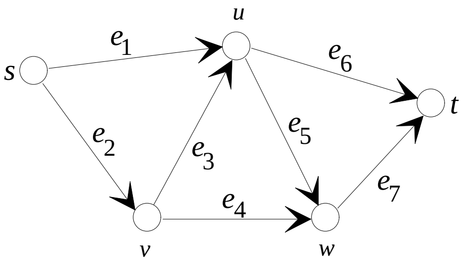
Numeric Example



$n = 53, \quad m = 203$

$R(G) = 0.986522, \ (53\ s) \qquad R_{st}(G) = 0.999984, \ (221\ s)$

André Pönitz: *Über eine Methode zur Konstruktion von Algorithmen für die Berechnung von Invarianten in endlichen ungerichteten Hypergraphen*, (About a generation method for algorithms for the calculation of invariants of finite graphs and hypergraphs), PhDThesis, Technische Universität Freiberg, 2004.



Algebraic Methods

$$R_{st}(G) = \bigoplus_{W \in \mathcal{W}} \bigotimes_{e \in W} p_e$$

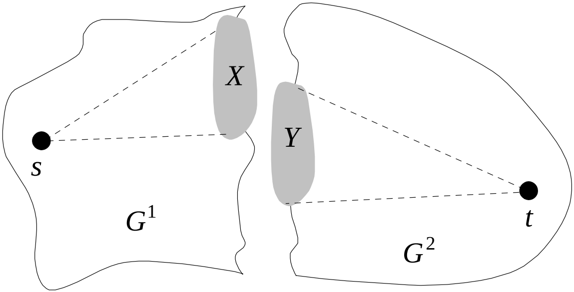
Iteration procedure

$$P = \begin{pmatrix} 0 & p_1 & p_2 & 0 & 0 \\ 0 & 0 & 0 & p_5 & p_6 \\ 0 & p_3 & 0 & p_4 & 0 \\ 0 & 0 & 0 & 0 & p_7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad s_i = \begin{cases} 1, & \text{if } i = s \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} \mathbf{x}_0 &= (0, \ldots, 0), \ \mathbf{s} = (s_1, \ \ldots, \ s_n) \\ \mathbf{x}_n &= \mathbf{x}_{n-1} \otimes P \oplus \mathbf{s} \text{ for } n > 0 \end{aligned}$$

Directed Graphs

Splitting



$$R_{st}(G) = \sum_{\emptyset \subset X \subseteq U} \sum_{Y \subseteq X} (-1)^{|X|-|Y|+1} R_{s,U\setminus Y}(G^1_{U\setminus Y}) R_{X,t}(G^2_X)$$